ELASTIC BUCKLING ANALYSIS OF SIMPLY SUPPORTED THIN PLATES USING THE DOUBLE FINITE FOURIER SINE INTEGRAL TRANSFORM METHOD

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ABSTRACT

In this work, the double finite Fourier sine integral transform method has been used to solve the buckling problem of thin rectangular plates with simply supported edges. The double finite Fourier sine integral transformation was applied to the governing partial differential equation of plates under in-plane compressive loads to reduce the problem to an algebraic eigenvalue – eigenvector problem, for the two cases of uniaxial compressive loading and biaxial compressive loading considered. The requirement for non trivial solutions was used to obtain the characteristic buckling equations for the two cases. The buckling equations were solved to obtain the critical buckling loads for uniaxial buckling and biaxial buckling. It was found that the expressions obtained were exactly identical with those given in literature sources which used Navier's methods and energy minimization methods.

Keywords: *Elastic buckling, finite Fourier sine integral transform method, thin plate, uniaxial buckling, biaxial buckling*

1. INTRODUCTION

Plates are extensively applied in many engineering structures such as aircraft wings, spacecraft panels, ship hulls and decks, building floor and roof, slabs, and offshore platform structures. Most plate structures, though capable of carrying tensile forces, are poor in resisting compressive forces (Yu, 2003). Usually buckling of compressed plates is a nonlinear phenomenon that takes place suddenly and may result in catastrophic structural failure. This underscores the importance of determining the buckling capacities of plates to avoid premature failures.

The first significant work on rectangular thin plate buckling was presented by Navier who, based on Kirchhoff's hypothesis, derived the stability equation of rectangular plates using method of the theory of elasticity (Navier, 1822). Since then, studies on the elastic and inelastic buckling of plates with various other types of shapes (circular, skew, quadrilateral, triangular, etc), boundary (clamped, free, etc) and loading conditions have been extensively reported in standard books, research reports and journal papers (Timoshenko and Gere 1961, Bulson 1970, Wang et al 2005, Xiang et al 12001, Batdorf and Houbolt 1946).

Plate buckling may be classified as elastic buckling and plastic (inelastic) buckling. In elastic buckling analysis, it is assumed that the critical buckling load is less than the elastic limit of the plate material. However, in practical problems, the plate might be stressed beyond the elastic limit before the onset of buckling, and the buckling problem becomes inelastic (plastic) buckling problem.

1.1 Research Aim and Objectives:

The aim of this study is to apply the double finite Fourier sine integral transform method to the elastic buckling analysis of simply supported thin plates under uniaxial and biaxial compressive loads. The objectives are as follows:

(i) to apply the double finite Fourier sine integral transformation to the governing partial differential equation of thin plates under uniaxial compressive load and under biaxial compressive load;
(ii) to show that the boundary value problem simplifies to an algebraic eigenvalue – eigenvector problem in terms of the transformed variable;
(iii) to solve the resulting algebraic eigenvalue – eigenvector problem and thus determine the buckling equations for the two cases of uniaxial compression in the *x*-direction, and biaxial compression in both the *x* and *y* directions;
(iv) to determine the buckling loads for the case of uniaxial compression, and for the case of biaxial compression for rectangular thin plates simply supported on all four edges.

2. LITERATURE REVIEW

2.1 Elastic Buckling of Rectangular Plates:

Navier (1822) derived the governing partial differential equation for stability analysis of rectangular thin plates under distributed transverse load by including the twisting action. The inclusion of the "twisting" term in the stability equation was particularly significant because the resistance of the rectangular thin plate to twisting can remarkably reduce transverse deflections under transverse distributed loads. Saint Venant (1883) modified the Navier's equation by including edge forces and shearing forces applied in the axial directions. Saint Venant's modified differential equation provided the groundwork for much of the experimental and theoretical studies on the elastic stability and elastic buckling of rectangular thin plates with various edge loads and edge support conditions. The basic form of the plate buckling problem is the problem of a simply supported rectangular thin plate under uniaxial compressive forces applied on the edges. Bryan (1891) first solved the problem by using the principle of minimization of the total potential energy functional to obtain the values of the critical buckling loads. He assumed that the buckling shape function for the problem of buckling of simply supported rectangular thin plates under uniaxial compression forces is a double Fourier sine series. Timoshenko (1925) assumed the buckling shape function as several sinusoidal half waves in the direction of the axial compression and used boundary (edge) support conditions to obtain a matrix which was solved to yield the critical (buckling) load. Timoshenko (1925) also studied the elastic buckling of uniaxially compressed rectangular plates (for uniform compressive force) for rectangular thin plates with two opposite edges simply supported and the other edges under different support conditions. He obtained results for buckling loads and buckled shapes that agreed with experimental results presented by Bridget et al (1934).

2.2 Solution Methods for Elastic Buckling Problems:

Generally, three broad methods are employed in the solution of the elastic buckling problems of plates. They are (i) the classical methods also called the equilibrium (Euler) methods, (ii) the variational methods (Energy methods), and (iii) the numerical methods. The classical methods aim to obtain closed form or mathematical solutions to the governing partial differential equations of equilibrium of the elastic buckling problem of plates within the plate domain, subject to the boundary conditions of loading and restraints of the plate edges. This result in an eigenvlaue eigenvector problem, which yields a characteristic equation called the buckling equation which roots yield the buckling loads. Mathematically rigorous methods for solving partial and ordinary differential equations subject to known boundary conditions fall into this category. The solutions obtained are called exact solutions, within the framework of the assumptions and the theory employed in constructing the governing equations. A major drawback of the classical methods is that they are difficult to apply to plates with fixed edges, free edges and mixed support conditions. Their application to plates with mixed support conditions lead to mathematical and analytical difficulties. The classical methods have been successfully applied to the elastic buckling problem of plates with simply supported edges for the case of uniaxial compressive load. Classical methods include Navier's double trigonometric series method, the separation of variables method, the Fourier series method, the methods of integral transformation etc. The variational (energy) methods are methods of elastic buckling analysis of plates based on the application of the minimum principle to the total potential energy of the plate buckling problem. They include the Ritz variational methold, Kantorovich variational method, Rayleigh-Ritz method, etc. The variational methods aim at minimizing the total potential energy functional of the elastic buckling problem of plates with respect to certain unknown generalised parameters of the triad displacement function to obtain the characteristic buckling equation, whose roots yield the buckling loads. The problem reduces to eigen-value-eigen-vector problem. Numerical methods of solving the elastic buckling problems of plates aim at obtaining approximate numerical solutions to the problem. They include the finite element methods, finite strip methods, boundary element methods, weighted residual methods, finite difference methods, improved finite difference methods, etc. The numerical formulation of the boundary value problem of plate buckling also yield eigenvalue-eigenvector problems. Ibearugbulem and Ezeh (2013) used the Taylor-Maclaurin's series shape function in the Ritz method to solve the buckling problem of axially compressed thin rectangular plate with clamped edges. Ezeh et al (2014) also applied the Galerkin's indirect variational method to the elastic buckling analysis of thin rectangular plates with all edges clamped for the case of uniform axial compression.

3. METHODOLOGY AND THEORETICAL FRAMEWORK

3.1 Double Finite Fourier Sine Integral Transform

The method of finite Fourier sine integral transformation was introduced by Doetsch (1935) for solving boundary value problems. It has been developed and generalized by several other researchers such as Kneitz (1938), Strandhagen (1944), Roettinger (1947) and Brown (1944). Mama et al (2017), Mama et al (2017) and Mama et al (2017) have successfully applied the finite Fourier sine transform method to the solution of general engineering problems. The double finite Fourier sine integral transform of $w(x, y)$, defined over the rectangular domain $0 \le x \le a, \quad 0 \le y \le b$ is denoted by $w(m, n)$ and given by:

$$w(m,n) = \int_0^a\int_0^b w(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \qquad (1)$$

where, $m = 1, 2, 3, \infty \qquad n = 1, 2, 3, \infty$

From this definition, the double finite sine integral transform of derivatives of $w(x, y)$ can be obtained, using integration by parts;

$$\int_0^a\int_0^b \frac{\partial^2 w}{\partial y^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$
$$= -\frac{n\pi}{b}\int_0^a\left((-1)^n w\big|_{y=b} - w\big|_{y=0}\right)\sin\frac{m\pi x}{a}dx - \left(\frac{n\pi}{b}\right)^2 w(m,n) \qquad (2)$$

$$\int_0^a\int_0^b \frac{\partial^4 w}{\partial x^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$
$$= \left(\frac{m\pi}{a}\right)^4 w(m,n) + \left(\frac{m\pi}{a}\right)^3\int_0^b\left((-1)^m w\big|_{x=a} - w\big|_{x=0}\right)\sin\frac{n\pi y}{b}dy$$
$$-\left(\frac{m\pi}{a}\right)\int_0^b\left((-1)^m \frac{\partial^2 w}{\partial x^2}\bigg|_{x=a} - \frac{\partial^2 w}{\partial x^2}\bigg|_{x=0}\right)\sin\frac{n\pi y}{b}dy \qquad (3)$$

$$\int_0^a\int_0^b \frac{\partial^4 w}{\partial x^2\partial y^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 w(m,n) \qquad (4)$$

For boundary value problems with Dirichlet boundary conditions,

$$w(x=0) = w(x=a) = 0 \qquad (5)$$

$$\frac{\partial^2 w}{\partial x^2}\bigg|_{x=0} = \frac{\partial^2 w}{\partial x^2}\bigg|_{x=a} = 0 \qquad (6)$$

$$w(y=0) = w(y=b) = 0 \qquad (7)$$

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=b} = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0} = 0 \qquad (8)$$

and the double finite sine integral transforms of the partial derivatives, Equations (2) – (3) simplify further to the following:

$$\int_0^a\int_0^b \frac{\partial^2 w}{\partial y^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = -\left(\frac{n\pi}{b}\right)^2 w(m,n) \qquad (9)$$

$$\int_0^a\int_0^b \frac{\partial^4 w}{\partial x^2 \partial y^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 w(m,n) \qquad (10)$$

$$\int_0^a\int_0^b \frac{\partial^4 w}{\partial x^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = \left(\frac{m\pi}{a}\right)^4 w(m,n) \qquad (11)$$

$$\int_0^a\int_0^b \frac{\partial^4 w}{\partial y^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = \left(\frac{n\pi}{a}\right)^4 w(m,n) \qquad (12)$$

3.2 Theoretical framework of rectangular thin plates under buckling

Thin plate theory was adopted in this study. The governing partial differential equation for thin rectangular plates under uniaxial compressive force N_x in the x-direction is: (Dima 2015, Bouazzar et al 2012, Iyengar 1988, Iberugbulem et al 2011, Chattopadhyay 2011)

$$D\nabla^4 w + N_x \frac{\partial^2 w}{\partial x^2} = 0 \qquad (13)$$

where $\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

∇^2 is the Laplace operator while ∇^4 is the biharmonic operator.

The equation for thin plates under biaxial compressive forces N_x, and N_y acting in the x and y directions respectively is:

$$D\nabla^4 w + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \qquad (14)$$

4. FINITE FOURIER SINE TRANSFORM METHOD FOR ELASTIC BUCKLING OF SIMPLY SUPPORTED PLATES UNDER UNIAXIAL COMPRESSION

The simply supported rectangular thin plate problem under uniform compressive force N_x as shown in Figure 1 is considered.

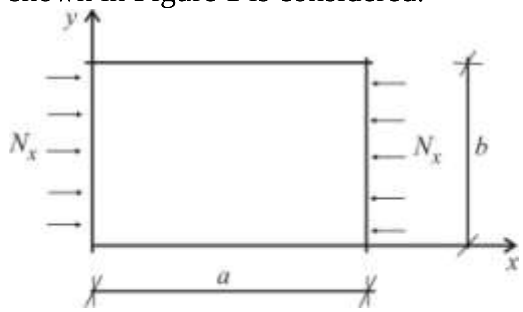

Figure 1: *Simply supported rectangular thin plate under uniform compressive force N_x*

Applying the double finite Fourier sine transformation, we have

$$\int_0^b\int_0^a\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = 0 \qquad (15)$$

$$\left(\frac{m\pi}{a}\right)^4\int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy + 2\times -\left(\frac{m\pi}{a}\right)^2 \times -\left(\frac{n\pi}{b}\right)^2\int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy + \left(\frac{n\pi}{b}\right)^4\int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$
$$+\frac{N_x}{D}\times -\left(\frac{m\pi}{a}\right)^2\int_0^b\int_0^a w(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = 0 \qquad (16)$$

Using Equation (1), we obtain:

$$\left(\frac{m\pi}{a}\right)^4 w(m,n) + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 w(m,n) + \left(\frac{n\pi}{b}\right)^4 w(m,n) - \left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}w(m,n) = 0 \qquad (17)$$

$$\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 - \left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}\right)w(m,n) = 0 \qquad (18)$$

$$\left(\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)^2 - \left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}\right)w(m,n) = 0 \qquad (19)$$

This is an algebraic eigenvalue – eigenvector problem. This is a system of homogeneous equations. For non-trivial solutions, $w(m,n) \neq 0$ hence the stability equation is given by the requirement that the coefficient matrix will vanish.

Thus,

$$\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)^2 - \left(\frac{m\pi}{a}\right)^2\frac{N_x}{D} = 0 \qquad (20)$$

Solving,

$$\left(\frac{m\pi}{a}\right)^2 \frac{N_x}{D} = \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)^2 \quad (21)$$

$$\frac{N_x}{D} = \left(\frac{a}{m\pi}\right)^2 \left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) \quad (22)$$

$$N_x = D\left(\frac{a}{m\pi}\right)^2 \left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) \quad (23)$$

$$= D\left(\left(\frac{m\pi}{a}\right)^2 + 2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{a}{m\pi}\right)^2\left(\frac{n\pi}{b}\right)^4\right) \quad (24)$$

$$= D\left(\left(\frac{m\pi}{a}\right)^2 + 2\left(\frac{n\pi}{b}\right)^2 + \frac{n^4}{m^2}\frac{a^2}{b^4}\pi^2\right) \quad (25)$$

$$N_x = \frac{D\pi^2}{a^2}\left(m^2 + 2\frac{a^2}{b^2}n^2 + \frac{n^4}{m^2}\frac{a^4}{b^4}\right) \quad (26)$$

Let $p = a/b$

$$N_x = \frac{D\pi^2}{a^2}\left(m^2 + 2p^2n^2 + \frac{n^4p^4}{m^2}\right) \quad (27)$$

$$N_x = \frac{D\pi^2}{b^2p^2}\left(m^2 + 2p^2n^2 + \frac{n^4p^4}{m^2}\right) \quad (28)$$

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{m^2}{p^2} + 2n^2 + \frac{n^4p^2}{m^2}\right) \quad (29)$$

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{m}{p} + \frac{n^2p}{m}\right)^2 = k\frac{D\pi^2}{b^2} \quad (30)$$

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{m^2}{p^2} + 2\frac{m}{p}\frac{n^2p}{m} + \frac{n^4p^2}{m^2}\right) \quad (31)$$

From the buckling load expression, it can be observed that as n increases, the critical load N_x also increases. Thus, for the lowest value of N_x, n must be equal to one. This means that the buckling mode shape is one half sine wave along the y coordinate direction. The lowest value of N_x can then be calculated using the calculus of minima and maxima.

$$\frac{dN_x}{dm} = \frac{2D\pi^2}{b^2}\left(\frac{m}{p} + \frac{p}{m}\right)\left(\frac{1}{p} - \frac{p}{m^2}\right) = 0 \quad (32)$$

This gives

$$m = p \quad (33)$$

Hence the lowest or critical buckling load is obtained as

$$N_{x_{cr}} = N_x(m = p, n = 1) \quad (34)$$

$$= \frac{D\pi^2}{b^2}\left(\frac{p}{p} + \frac{1p}{p}\right)^2 = \frac{4D\pi^2}{b^2} \quad (35)$$

Hence, a Kirchhoff rectangular plate simply supported along the four edges $x = 0$, $x = a$, $y = 0$, and $y = b$ and subjected to uniform compressive force in the x-direction buckles with one half wave in the y-direction and p half waves in the x-direction i.e. p must be an integer. This implies that the rectangular plate buckles into square plates. For non-integer values of p, the buckling load is observed to be higher than that for integer values. For non integer values the critical buckling load $(N_x)_{cr}$ can be expressed as

$$(N_x)_{cr} = \frac{kD\pi^2}{b^2} \quad (36)$$

where $k = \left(\frac{m}{p} + \frac{n^2p}{m}\right)^2 \quad (37)$

k is called the buckling or stability coefficient. The critical stress σ_{xx} per unit length is found as:

$$(\sigma_{xx})_{cr} = \frac{(N_x)_{cr}}{t} = \frac{kE}{12(1-\mu^2)}\left(\frac{t}{b}\right)^2 \quad (38)$$

$$= \frac{k\pi^2}{tb^2}\cdot\frac{Et^3}{12(1-\mu^2)} \quad (39)$$

$$= \frac{k\pi^2}{b^2}\cdot\frac{Et^2}{12(1-\mu^2)} \quad (40)$$

$$(\sigma_{xx})_{cr} = k\frac{\pi^2E}{12(1-\mu^2)}\left(\frac{t}{b}\right)^2 \quad (41)$$

$$(\sigma_{xx})_{cr} = K\,E\left(\frac{t}{b}\right)^2 \quad (42)$$

where $K = \frac{k\pi^2}{12(1-\mu^2)} \quad (43)$

The expression for the critical compressive stress $\sigma_{xx_{cr}}$ in thin plates is called the Bryan equation, after the English naval engineer who derived it from first principles using the total potential energy minimization method to calculate the buckling loads of thin rectangular plates in the hulls of steel ships.

Table 1: Stability coefficients of uniaxially compressed rectangular thin plates with simply supported edges

a/b	K (Iyengar, 1988)	K (Present study)
0.1	102.01	102.01
0.2	27.04	27.04
0.3	13.2011	13.2011
0.4	8.41	8.41
0.5	6.25	6.25
0.6	5.1378	5.1378
0.7	4.5308	4.5308
0.8	4.2025	4.2025
0.9	4.0446	4.0446
1.0	4	4

For long plates, $a/b \geq 3$, $\mu = 0.3$, K = 4.0, and

$$\sigma_{xx_{cr}} = 3.62E\frac{t^2}{b^2} \quad (44)$$

For small values of a/b, i.e. $a/b \leq 1$, when $m = 1$, the buckling coefficient simplifies to:

$$k = \left(\frac{b}{a}\right)^2\left(1+\left(a/b\right)^2\right)^2 \quad (45)$$

Then, when $\left(a/b\right)^2 << 1$, the buckling coefficient simplifies further to:

$$k ; \left(b/a\right)^2 \quad (46)$$

Thus for $\left(a/b\right)^2 << 1$, the critical stress becomes

$$\sigma_{xx_{cr}} = \frac{\pi^2 E}{12(1-\mu^2)}\frac{b^2}{a^2}\cdot\frac{t^2}{b^2} \quad (47)$$

$$\sigma_{xx_{cr}} = \frac{\pi^2 E}{12(1-\mu^2)}\left(\frac{t^2}{a^2}\right) \quad (48)$$

5. FINITE FOURIER SINE INTEGRAL TRANSFORM METHOD FOR THE ELASTIC BUCKLING OF SIMPLY SUPPORTED RECTANGULAR THIN PLATES UNDER BIAXIAL COMPRESSION

A rectangular thin plate simply supported on all four edges $x = 0$, $x = a$, $y = 0$, $y = b$ and subject to biaxial compressive loads N_x and N_y in the x and y directions as shown in Figure 2 is considered.

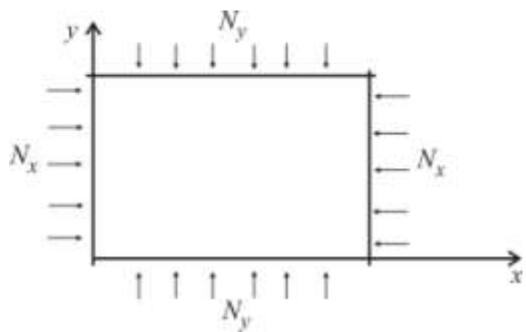

Figure 2: *Simply supported rectangular thin plate under biaxial compressive forces.*

Applying the double finite Fourier sine integral transformation to the governing plate equation, we obtain:

$$\int_0^a\int_0^b\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{N_x}{D}\frac{\partial^2 w}{\partial x^2}\right.$$

$$\left.+\frac{N_y}{D}\frac{\partial^4 w}{\partial y^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = 0 \quad (49)$$

The linearity property of the finite Fourier sine integral transformation allows us to write:

$$\int_0^b\int_0^a\frac{\partial^4 w}{\partial x^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy + 2\int_0^b\int_0^a\frac{\partial^4 w}{\partial x^2\partial y^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$+\int_0^b\int_0^a\frac{\partial^4 w}{\partial y^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy + \frac{N_x}{D}\int_0^b\int_0^a\frac{\partial^2 w}{\partial x^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$+\frac{N_y}{D}\int_0^b\int_0^a\frac{\partial^2 w}{\partial y^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = 0 \quad (50)$$

$$\left(\frac{m\pi}{a}\right)^4 \int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$
$$+\,2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$
$$+\left(\frac{n\pi}{b}\right)^4 \int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$
$$-\frac{N_x}{D}\left(\frac{m\pi}{a}\right)^2 \int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$
$$-\frac{N_y}{D}\left(\frac{n\pi}{b}\right)^2 \int_0^b\int_0^a w\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = 0 \qquad (51)$$

$$\left(\frac{m\pi}{a}\right)^4 w(m,n)+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 w(m,n)+\left(\frac{n\pi}{a}\right)^4 w(m,n)$$
$$-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}w(m,n)-\left(\frac{n\pi}{b}\right)^2\frac{N_y}{b}w(m,n)=0 \qquad (52)$$

$$\left(\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}-\left(\frac{n\pi}{b}\right)^2\frac{N_y}{D}\right)w(m,n)=0 \qquad (53)$$

We thus obtain an algebraic eigenvalue – eigenvector problem. This is a system of homogeneous algebraic equations. For non-trivial solutions, $w(m,n)\neq 0$, and the stability equation is given by the requirement that the coefficient matrix will vanish. Thus, the stability equation is obtained as:

$$\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}-\left(\frac{n\pi}{b}\right)^2\frac{N_y}{D}=0 \qquad (54)$$

Solving,

$$\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}+\left(\frac{n\pi}{b}\right)^2\frac{N_y}{D}=\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2 \qquad (55)$$

This is the same equation obtained by Dima (2015) and Bouazzar et al (2012)

Let $N_x = N_0$, $N_y=\gamma N_0$; $\gamma=\frac{N_y}{N_x}$

$$\left(\frac{m\pi}{a}\right)^2 N_0+\left(\frac{n\pi}{b}\right)^2\gamma N_0 = D\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2 \qquad (56)$$

$$N_0\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\gamma\right)=D\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2 \qquad (57)$$

$$N_0(m,n)=\frac{D\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2}{\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\gamma\right)} \qquad (58)$$

Let $s = a/b \Rightarrow a = sb$

$$N_0(m,n)=\frac{D\left(\frac{m^4\pi^4}{s^4b^4}+2\frac{m^2\pi^2}{s^2b^2}\frac{n^2\pi^2}{b^2}+\frac{n^4\pi^4}{b^4}\right)}{\left(\frac{m^2\pi^2}{s^2b^2}+\frac{n^2\pi^2}{b^2}\gamma\right)} \qquad (59)$$

$$N_0(m,n)=\frac{D\left(\frac{m^4\pi^4+2m^2\pi^4n^2s^2+s^4n^4\pi^4}{s^4b^4}\right)}{\left(\frac{m^2\pi^2+s^2n^2\pi^2\gamma}{s^2b^2}\right)} \qquad (60)$$

$$N_0(m,n)=D\left(\frac{m^4\pi^4+2m^2\pi^4n^2s^2+s^4n^4\pi^4}{s^4 s^2\ b^4 b^2}\frac{s^2 b^2}{m^2\pi^2+s^2n^2\pi^2\gamma}\right) \qquad (61)$$

$$N_0(m,n)=\frac{D\pi^2}{b^2}\left(\frac{m^4+2m^2n^2s^2+s^4n^4}{m^2s^2+n^2s^4\gamma}\right) \qquad (62)$$

$$N_0(m,n)=\frac{D\pi^2}{s^2b^2}\left(\frac{m^4+2m^2n^2s^2+s^4n^4}{m^2+n^2s^2\gamma}\right) \qquad (63)$$

$$N_0(m,n)=\frac{D\pi^2}{a^2}\left(\frac{(m^2+n^2s^2)^2}{(m^2+n^2s^2\gamma)}\right) \qquad (64)$$

$$N_0(m,n)=\frac{D\pi^2}{a^2}K_1 \qquad (65)$$

$$N_0(m,n)=\frac{D\pi^2}{b^2}K_2 \qquad (66)$$

$$K_2=\frac{m^4+2m^2n^2s^2+n^4s^4}{m^2s^2+n^2s^2\gamma} \qquad (67)$$

$$K_1 = \frac{(m^2 + n^2 s^2)^2}{m^2 + n^2 s^2 \gamma} \quad (68)$$

For $p = b/a$,

$$N_0(m,n) = \frac{D\pi^2}{b^2}\left(\frac{p^4 m^4 + 2p^2 m^2 n^2 + n^4}{p^2 m^2 + \gamma n^2}\right) \quad (69)$$

The critical buckling load is the smallest value of $N_0(m, n)$. For a particular simply supported rectangular thin plate under biaxial uniform compression forces N_x, N_y, the critical buckling load is seen from Equation (69) to be dependent upon the values of m, n, γ and the geometric and elastic properties of the plate.

5.1 Biaxial Buckling of Square Plates:

For square thin plates subject to the same magnitude of uniform compressive forces on both edges, i.e. biaxial compression with $\gamma = 1$, we obtain from Equation (69):

$$N_0(m,n) = \frac{D\pi^2}{a^2}\left(\frac{m^4 + 2m^2 n^2 + n^4}{m^2 + n^2}\right) \quad (70)$$

$$N_0(m=1,n) = \frac{D\pi^2}{a^2}\left(\frac{1 + 2n^2 + n^4}{1 + n^2}\right) \quad (71)$$

For rectangular plates, where $\gamma = 1$,

$$N_0(\mathrm{m},n) = \frac{\pi^2 D}{b^2}\left(m^2 p^2 + n^2\right) \quad (72)$$

The critical buckling load occurs when $m = n = 1$ and

$$N_{xx_{cr}} = (1 + p^2)\frac{D\pi^2}{b^2} = k\frac{D\pi^2}{b^2} \quad (73)$$

$$k = 1 + p^2 \quad (74)$$

For square plates, $p = 1$,

$$N_{xx_{cr}} = \frac{2D\pi^2}{b^2} \quad (75)$$

6. DISCUSSIONS

In this study, the double finite Fourier sine integral transform method has been successfully applied to solve the buckling problem of simply supported rectangular thin plate under uniform uniaxial and biaxial compressive forces. The double finite Fourier sine transformation was applied to both sides of the governing partial differential equation, for the two cases of uniform axial compression force N_x in the x-direction and uniform biaxial compression forces N_x and N_y in the x- and y-directions respectively. Consequently, the fourth order linear partial differential equation was found, upon application of the linearity property of the finite Fourier sine integral transformation and integration by parts, to simplify to the algebraic eigenvalue – eigenvector problems presented as Equations (18) or (19) for the case of the simply supported rectangular thin plate under uniform uniaxial compression, and Equation (53) for the case of the simply supported rectangular thin plate under uniform biaxial compression. The stability equations were then obtained for non trivial solutions, by enforcing the requirement that $w(m,n) \neq 0$ and then the determinant of the coefficient matrix should be zero. The stability or characteristic buckling equations were thus found for the case of simply supported thin rectangular plate under uniaxial compression N_x as Equation (20). This was solved to determine the buckling load as Equation (26). Equation (26) reveals that the buckling load depends upon the geometrical shape of the rectangular plate as measured by the in-plane dimensions a and b; the integer parameters m and n, and the flexural rigidity of the plate material D, which in turn depends upon the plate thickness, t, the plate modulus of elasticity, E, and the Poisson's ratio, μ. The buckling load was further presented in terms of the buckling coefficient, k, as Equation (38), for simply supported rectangular thin plate under uniaxial compression. The critical buckling load is obtained using the calculus of maxima and minima on the expression for the buckling load; and this was found as Equation (35). The critical buckling stress was found as Equation

(41). The buckling coefficient for simply supported thin rectangular plates under uniaxial compression N_x as shown in Table 1 was found to be exactly the same as the expression obtained by Iyengar (1988), Timonshenko (1925), Bryan (1891) and Navier (1822), who used other solutions methods like the Navier series method, and total potential energy minimization methods to solve the same problem. For the case of simply supported thin rectangular plates under uniform biaxial compression, N_x and N_y, the buckling load was found for the case where the biaxial compression forces are related by $N_y = \gamma N_x = \gamma N_0$ where $\gamma = N_y / N_x$ as Equation (38). The solution is also presented in terms of the plate aspect ratios, $s = a/b$ or $p = b/a$ as Equations (64) and (69). This equation was particularized to the problem of square thin plates subject to the same magnitude of uniform compressive forces on both edges, and the buckling load obtained generally as Equation (70). The critical buckling load was obtained for simply supported rectangular thin plates under the same magnitude of uniform compressive force as Equation (72) and the critical buckling load obtained from the calculus of maxima and minima applied to Equation (72) as Equation (73). For square thin plates simply supported on the four edges and subject to biaxial compression forces of the same magnitude, the critical buckling load was obtained as Equation (75).

7. CONCLUSIONS

From this work, the following conclusions can be made:

(i) the double finite Fourier sine integral transformation, being a linear transformation is ideally suitable for the solution of the linear partial differential equation of fourth order that governs simply supported thin rectangular plates under uniform uniaxial and biaxial compression forces on the edges of the plate

(ii) the double Fourier sine integral transformation is appropriate for the problem since the simply supported edges satisfy the Dirichlet boundary conditions, which offer simplifications to the applications of the method

(iii) the application of the finite Fourier sine integral transformation simplifies the problem of simply supported thin plate buckling from a differential equation to an algebraic eigenvalue equation

(iv) the characteristic buckling equations obtained for both cases are the same as the characteristic buckling equations obtained by other researchers who used Navier's series and energy minimization methods

(v) the critical buckling loads obtained were also the same as the critical buckling loads obtained in literature by other scholars who used Navier's series and energy minimization methods.

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