BENDING ANALYSIS OF SIMPLY SUPPORTED RECTANGULAR KIRCHHOFF PLATES UNDER LINEARLY DISTRIBUTED TRANSVERSE LOAD

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ABSTRACT

In this study, the Kantorovich-Vlasov method was applied to the flexural analysis of simply supported rectangular Kirchhoff plates under hydrostatic load distribution over the entire plate domain. Vlasov method was used to obtain the shape function in the *x*-direction, and Kantorovich method was used to choose the shape function for the plate in series form; as the product of unknown functions $g_m(y)$ and the shape functions in the *x*-direction. The total potential energy functional Π was then obtained, and minimized using Euler-Lagrange differential equations to obtain the unknown functions ($g_m(y)$). Enforcement of boundary conditions in the *y*-direction and the demands of symmetry led to the determination of integration constants. Bending moments were obtained using the bending moment-deflection equations. Deflections and bending moments at the center of the plate were then obtained in terms of the aspect ratios. The results for deflection, and bending moments at the center of the plate were found to be the same as results obtained by Timoshenko and Woinowsky-Krieger who used the Levy method.

Keywords: *Kantorovich-Vlasov method, Kirchhoff plate, hydrostatic load distribution*

1. INTRODUCTION

Plates are three dimensional structures with one dimension that is much smaller than the other two inplane dimensions (Chandrashekhara, 2011; Szilard, 2004; Timoshenko and Woinowsky-Krieger, 1959). They have extensive applications in civil, mechanical, aeronautical, naval and geotechnical engineering in the modeling of ship hulls, roof and floor slabs, retaining walls, foundation slabs, etc. Plates are classified as thin plates, moderately thick plates and thick plates based on the ratios of the thickness (h) to the least lateral dimension, (a). They are also classified according to the geometry as rectangular, square, elliptical, skew, circular, rhombic, etc. They are also described based on their material properties as homogeneous, heterogeneous, anisotropic, isotropic, and orthotropic. Several theories, and models have been used to describe plates behaviour. They include: Kirchhoff plate theory, also called the Kirchhoff-Love plate theory or the classical plate theory, Mindlin (1951) plate theory (Ike, 2017). Reissner (1945) plate theory, Levy plate theory, Refined plate theories (Shimpi, 2007; Suetake, 2006). Kirchhoff plate theory is effective for thin plates and is adopted in this study.

2. METHODS OF ANALYSIS OF PLATES

Two basic methods are used to solve the boundary value problems of plates. They are broadly – closed form analytical methods and numerical or approximate methods. Closed form analytical methods are mathematical methods that seek to obtain solutions to the plate problem that satisfy the governing equations at all points on the plate region as well as on the plate boundaries. They include: Navier (1823) series method, Levy (1899) series method separation of variables method, eigen function expansion methods, and integral transform methods (Mama et al, 2017). They have been used to obtain solutions to plate bending problems for different edge support conditions and different loading conditions (Chandrashekhara, 2011; Szilard, 2004; Kapadiya and Patel, 2015; Timoshenko and Woinowsky-Krieger 1959). Numerical methods seek to obtain approximate solutions to the plate problem, and are used in complicated problems where closed form solutions are difficult to obtain. Numerical methods include: finite element methods, boundary element methods, finite difference methods (Kapadiya and Patel, 2015), Variational Ritz methods (Aginam et al, 2012), Variational Galerkin methods (Osadebe et al, 2016; Nwoji et al, 2017), Variational Kantorovich methods (Nwoji et al, 2017; Ike, 2017), Weighted residual methods, Bubnov-Galerkin methods, and Collocation methods. Numerical methods have been extensively used to solve plate problems for different types of loading and edge support conditions (Eze et al, 2013; Aginam et al, 2012; Osadebe et al, 2016; Nwoji et al 2017; Ike, 2017).

3. RESEARCH AIM AND OBJECTIVES

The aim of this study is to apply the Kantorovich-Vlasov method to the analysis of simply supported rectangular Kirchhoff plates under hydrostatic load distribution over the entire plate region. The specific objectives are:

(i) to use the Vlasov method to derive suitable displacement shape functions for the simply supported plate in the *x* coordinate direction

(ii) to obtain the total potential energy functional Π for the plate flexure problem considered

(iii) to obtain the Euler-Lagrange differential equations for the extremization of the total potential energy functional derived

(iv) to solve the Euler-Lagrange differential equations subject to the boundary conditions of the plate edges in the *y*-direction

(v) to obtain the bending moment expressions

(vi) to obtain the values of the deflection and bending moments at the center of the plate for various plate aspect ratios

4. APPLICATION OF KANTOROVICH-VLASOV METHOD

Consider the simply supported rectangular Kirchhoff plate under linearly distributed (hydrostatic) load of intensity $p(x) = p_0x/a$ as shown in Figure 1.

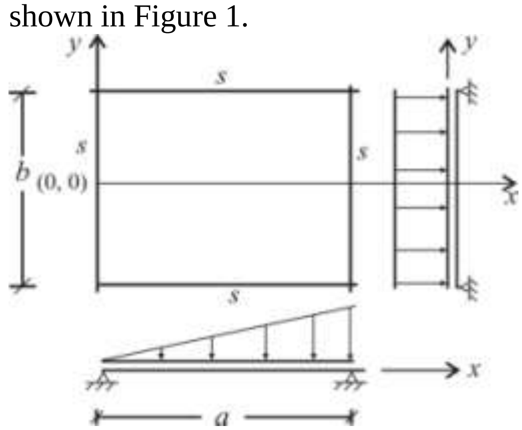

Figure 1: *Rectangular Kirchhoff plate under hydrostatic load*

The system of Cartesian coordinate axes is chosen as shown in Figure 1 in order to take advantage of the symmetrical nature of the problem in the *y* coordinate direction. The plate is subjected to a transverse distributed load of intensity $p(x) = p_0 x/a$ over the plate domain, $0 \le x \le a, \quad -b/2 \le y \le b/2$. The displacement (basis) shape function is chosen in the *x*-

direction using the Vlasov method, as the eigen function of a harmonically vibrating prismatic Euler-Bernoulli beam with identical end support conditions as the plate in the *x* direction.

4.1 Displacement Shape Function in the *x*-Direction:

Following Vlasov method, the displacement shape function of the Kirchhoff plate is chosen as the displacement shape function of a prismatic Bernoulli-Euler beam with identical end conditions under natural vibrations. For free vibrations of prismatic Euler-Bernoulli beams with simply supported ends at $x = 0$, $x = a$, the governing equation is

$$EI\frac{\partial^4 X(x,t)}{\partial x^4} + m\frac{\partial^2 X(x,t)}{\partial t^2} = 0 \quad (1)$$

Subject to the boundary conditions,

$$X(x=0,t) = 0 = X''(x=0,t) \quad (2)$$

$$X(x=a,t) = 0 = X''(x=a,t) \quad (3)$$

where $X(x, t)$ is the dynamic displacement, x is the space variable in the longitudinal axis of the beam, t is time. EI is modulus of rigidity of the beam, I is moment of inertia, E is Young's modulus of elasticity, m is the mass per unit length of the beam.

For free harmonic vibrations,

$$X(x,t) = f(x)\varphi(t) = f(x)\cos\omega t \quad (4)$$

where $\varphi(t)$ is the harmonic function expressing dependence of X on time, and $f(x)$ is the modal displacement function.

Substituting Equation (4) into Equation (1) yields:

$$\left(f^{iv}(x) - \frac{m\omega_n^2}{EI}f(x)\right)\cos\omega_n t = 0 \quad (5)$$

For non-trivial solutions, $\cos\omega_n t \neq 0$ (6)

The characteristic equation then becomes:

$$f^{iv}(x) - \frac{m\omega_n^2}{EI}f(x) = 0 \quad (7)$$

Let $\frac{m\omega_n^2}{EI} = \frac{\beta^4}{a^4}$ (8)

Then Equation (7) becomes:

$$f^{iv}(x) - \frac{\beta^4}{a^4}f(x) = 0 \quad (9)$$

The general solution of Equation (9) is:

$$f(x) = c_1\sin\frac{\beta x}{a} + c_2\cos\frac{\beta x}{a} + c_3\sinh\frac{\beta x}{a} + c_4\cosh\frac{\beta x}{a} \quad (10)$$

where c_1, c_2, c_3 and c_4 are the four constants of integration which can be obtained from the boundary conditions.

For the *m*th vibration, the eigen functions are:

$$f_m(x) = c_{1_m}\sin\frac{\beta_m x}{a} + c_{2_m}\cos\frac{\beta_m x}{a} + c_{3_m}\sinh\frac{\beta_m x}{a} + c_{4_m}\cosh\frac{\beta_m x}{a} \quad (11)$$

For simple supports at the edges $x = 0$, $x = a$, the boundary conditions are:

$$f_m(x=0) = 0 \qquad f_m''(x=0) = 0 \quad (12)$$

$$f_m(x=a) = 0 \qquad f_m''(x=a) = 0 \quad (13)$$

Differentiating Equation (11) twice with respect to x, we obtain:

$$f_m''(x) = \frac{\beta_m^2}{a^2}\left(-c_{1_m}\sin\frac{\beta_m x}{a} - c_{2_m}\cos\frac{\beta_m x}{a} + c_{3m}\sinh\frac{\beta_m x}{a} + c_{4_m}\cosh\frac{\beta_m x}{a}\right) \quad (14)$$

Applying the boundary conditions in Equation (12) and (13), we obtain:

$$f(0) = c_{2_m} + c_{4_m} = 0 \quad (15)$$

$$f''(0) = -c_{2_m} + c_{4_m} = 0 \quad (16)$$

$$f(a) = c_{1_m}\sin\beta_m + c_{2_m}\cos\beta_m + c_{3_m}\sinh\beta_m + c_{4_m}\cosh\beta_m = 0 \quad (17)$$

$$f''(a) = \frac{\beta_m^2}{a^2}\left(-c_{1_m}\sin\beta_m - c_{2_m}\cos\beta_m + c_{3_m}\sinh\beta_m + c_{4_m}\cosh\beta_m\right) = 0 \quad (18)$$

Solving Equations (15), (16), (17) and (18), we obtain:

$$c_{2_m} = 0 \quad (19)$$

$$c_{4_m} = 0 \quad (20)$$

$$c_{3_m} = 0 \quad (21)$$

Thus,

$$f_m(x) = c_{1_m}\sin\frac{\beta_m x}{a} \quad (22)$$

Then $f_m(x=a) = 0$ would lead to:

$c_{1_m}\sin\beta_m = 0$ (23)

For non trivial solutions,

$$c_{1_m} \neq 0 \quad (24)$$

$$\therefore \sin\beta_m = 0 \quad (25)$$

$$\beta_m = \sin^{-1}0 = m\pi \quad (26)$$

$m = 1, 2, 3,$

Then Equation (22) becomes:

$$f_m(x) = c_{1_m} \sin \frac{m\pi x}{a} \quad (27)$$

The eigen functions for simply supported prismatic Euler-Bernoulli beams thus gives the displacement shape function in the *x*-direction:

$$f_m(x)(\text{shape function}) = \sin \frac{m\pi x}{a} \quad (28)$$

4.2 Total Potential Energy Functional:

The total potential energy functional Π for a simply supported rectangular Kirchhoff plate under transverse distributed load $p(x)$ is given by;

$$\Pi = \frac{D}{2}\iint_{R^2} (\nabla^2 w)^2 \, dxdy - \iint_{R^2} pw \, dxdy \quad (29)$$

where w is the deflection of the plate middle surface, D is the flexural rigidity of the plate, and ∇^2 is the Laplacian given by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} \quad (30)$$

Following the Kantorovich method, the plate deflection is considered as:

$$w(x, y) = \sum_{m=1}^{\infty} g_m(y) \sin \frac{m\pi x}{a} \quad (31)$$

where $g(y)$ is an unknown function of y which we seek to determine.

Substituting Equation (31) into Equation (29), we obtain;

$$\Pi = \frac{D}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_0^a \sum_{m=1}^{\infty}\left(\left(g_m''(y) - \left(\frac{m\pi}{a}\right)^2 g_m(y)\right)\sin\frac{m\pi x}{a}\right)^2 dxdy$$

$$-\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_0^a \left(\sum_{m=1}^{\infty} g_m(y)\sin\frac{m\pi x}{a}\right)\frac{p_0 x}{a}\,dxdy \quad (32)$$

$$\Pi = \frac{D}{2}\sum_{m=1}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_0^a \left[(g_m''(y))^2 - 2\left(\frac{m\pi}{a}\right)^2 g_m(y)g_m''(y)\right.$$

$$\left. + \left(\frac{m\pi}{a}\right)^4 (g_m(y))^2\right]\sin^2\frac{m\pi x}{a}\,dxdy - \sum_{m=1}^{\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_0^a \frac{p_0 x}{a}\sin\frac{m\pi x}{a}\,g_m(y)\,dxdy \quad (33)$$

Simplifying,

$$\Pi = \sum_{m=1}^{\infty}\frac{Da}{4}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[(g_m''(y))^2 - 2\left(\frac{m\pi}{a}\right)^2 g_m(y)g_m''(y)\right.$$

$$\left. + \left(\frac{m\pi}{a}\right)^4 (g_m(y))^2\right]dy - \sum_{m=1}^{\infty}\frac{p_0 a}{m\pi}(-1)^{m+1}\int_{-\frac{b}{2}}^{\frac{b}{2}} g_m(y)\,dy \quad (34)$$

or

$$\Pi = \sum_{m=1}^{\infty}\frac{Da}{4}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[(g_m''(y))^2 - 2\left(\frac{m\pi}{a}\right)^2 g_m''(y)g_m(y)\right.$$

$$\left. + \left(\frac{m\pi}{a}\right)^4 g_m(y) - \frac{4p_0(-1)^{m+1}}{m\pi D} g_m(y)\right]dy \quad (35)$$

$$\Pi = f\left(g_m(y),\ g_m''(y),\ y\right) \quad (36)$$

4.3 Euler-Lagrange Differential Equation:

The Euler-Lagrange differential equation which gives the condition for the extremization of the functional Π is given by:

$$\frac{\partial F}{\partial g_m} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial g_m'}\right) + \frac{d^2}{dy^2}\left(\frac{\partial F}{\partial g_m''}\right) = 0 \quad (37)$$

where F is the integrand in the expression for Π.

$$F = (g_m''(y))^2 - 2\left(\frac{m\pi}{a}\right)^2 g_m(y)g_m''(y) + \left(\frac{m\pi}{a}\right)^4 (g_m(y))^2 - \frac{4p_0(-1)^{m+1}}{m\pi D} g_m(y) \quad (38)$$

Performing the differentiations, the Euler-Lagrange differential equation of equilibrium becomes the fourth order linear ordinary differential equation in $g_m(y)$:

$$g_m^{iv}(y) - 2\left(\frac{m\pi}{a}\right)^2 g_m''(y) + \left(\frac{m\pi}{a}\right)^4 g_m(y) = \frac{2p_0(-1)^{m+1}}{m\pi D} \quad (39)$$

4.4 Solution of the Euler-Lagrange Differential Equation:

For the homogeneous solution,

$$g_m^{iv}(y) - 2\left(\frac{m\pi}{a}\right)^2 g_m''(y) + \left(\frac{m\pi}{a}\right)^4 g_m(y) = 0 \quad (40)$$

Let the homogeneous solution $g_h(y)$ be in the form:

$$g_h(y) = e^{k_m y} \quad (41)$$

Then,

$$\left(k_m^4 - 2\left(\frac{m\pi}{a}\right)^2 k_m^2 + \left(\frac{m\pi}{a}\right)^4\right)e^{k_m y} = 0 \quad (42)$$

For non trivial solutions,

$$e^{k_m y} \neq 0 \quad (43)$$

The auxiliary (characteristic) polynomial becomes:

$$k_m^4 - 2\left(\frac{m\pi}{a}\right)k_m^2 + \left(\frac{m\pi}{a}\right)^4 = 0 \quad (44)$$

The roots are:

$$k_m = +\frac{m\pi}{a} \quad \text{(twice)} \quad (45)$$

$$k_m = -\left(\frac{m\pi}{a}\right) \quad \text{(twice)} \quad (46)$$

The homogeneous solution becomes

$$g_h(y) = A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + C_m \frac{m\pi y}{a}\cosh\frac{m\pi y}{a} + D_m \sinh\frac{m\pi y}{a} \quad (47)$$

The Kirchhoff plate bending problem is symmetric in the *y*-direction, and it is expected that the deflection $g_h(y)$ be symmetrical. Hence, for symmetry,

$$C_m = 0 \quad (48)$$

$$D_m = 0 \quad (49)$$

and the homogeneous solution becomes

$$g_h(y) = A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} \quad (50)$$

The particular solution $g_p(y)$ should satisfy the non-homogeneous Euler-Lagrange differential equation of equilibrium (Equation (39)).

Thus,

$$g_p^{iv}(y) - 2\left(\frac{m\pi}{a}\right)^2 g_p''(y) + \left(\frac{m\pi}{a}\right)^2 g_p(y) = \frac{2p_0(-1)^{m+1}}{m\pi D} \quad (51)$$

The applied hydrostatic load is constant along the *y* direction. Hence, the particular solution is expected to be constant in the *y* direction.

Thus, $g_p''(y) = g_p^{iv} = 0$ (52)

Then Equation (51) becomes:

$$\left(\frac{m\pi}{a}\right)^4 g_p(y) = \frac{2p_0(-1)^{m+1}}{m\pi D} \quad (53)$$

$$g_p(y) = \frac{2p_0 a^4(-1)^{m+1}}{(m\pi)^5 D} \quad (54)$$

4.5 General solution:

The general solution, by the principle of linearity is the sum of the homogeneous and

particular solutions, and is given by:

$$w(x,y) = \sum_{m=1}^{\infty}\left(A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + \frac{2p_0 a^4}{\pi^5 D}\frac{(-1)^{m+1}}{m^5}\right)\sin\frac{m\pi x}{a} \quad (55)$$

where A_m and B_m are constants that are determined using the boundary conditions of the plate on the edges $y = \pm b/2$.

4.6 Enforcement of Boundary: Conditions at $y = \pm b/2$

The boundary conditions on the edges $y = \pm b/2$ are

$$w\left(x,\ y = \pm b/2\right) = 0 \quad (56)$$

$$\frac{\partial^2 w}{\partial y^2}\left(x,\ y = \pm b/2\right) = 0 \quad (57)$$

From condition in Equation (56), we have:

$$A_m \cosh\frac{m\pi b}{2a} + B_m \frac{m\pi b}{2a}\sinh\frac{m\pi b}{2a} + \frac{2p_0 a^4(-1)^{m+1}}{(m\pi)^5 D} = 0 \quad (58)$$

Using Equation (57), we obtain:

$$A_m \cosh\frac{m\pi y}{2a} + B_m\left(2\cosh\frac{m\pi b}{2a} + \frac{m\pi b}{2a}\sinh\frac{m\pi b}{2a}\right) = 0 \quad (59)$$

From Equation (59), we write:

$$A_m = -B_m\left(2 + \frac{m\pi b}{2a}\tanh\frac{m\pi b}{2a}\right) \quad (60)$$

Substituting Equation (60) into Equation (58), we obtain:

$$-B_m\left(2 + \frac{m\pi b}{2a}\tanh\frac{m\pi b}{2a}\right) + B_m \frac{m\pi b}{2a}\sinh\frac{m\pi b}{2a} = -\frac{2p_0 a^4(-1)^{m+1}}{(m\pi)^5 D} \quad (61)$$

Simplifying Equation (61),

$$B_m = \frac{(-1)^{m+1}}{(m\pi)^5 \cosh \frac{m\pi b}{2a}} \frac{p_0 a^4}{D} \qquad (62)$$

Using Equation (60),

$$A_m = -\frac{\left(2 + \frac{m\pi b}{2a} \tanh \frac{m\pi b}{2a}\right)(-1)^{m+1}}{(m\pi)^5 \cosh \frac{m\pi b}{2a}} \frac{p_0 a^4}{D} \qquad (63)$$

From Equation (31), the deflection is given by

$$w(x, y) = \sum_{m=1}^{\infty} \left(A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} + \frac{2 p_0 a^4}{\pi^5 D} \frac{(-1)^{m+1}}{m^5} \right) \sin \frac{m\pi x}{a} \qquad (64)$$

where A_m and B_m are given by Equations (63) and (62) respectively.

4.7 Deflection along the *x*-axis:

The deflection along the *x*-axis, ($y = 0$) is given from Equation (64), by:

$$w(x,\ y = 0) = \sum_{m=1}^{\infty} \left(A_m + \frac{2 p_0 a^4}{D\pi^5} \frac{(-1)^{m+1}}{m^5} \right) \sin \frac{m\pi x}{a} \qquad (65)$$

$$= \frac{p_0 a^4}{D} \sum_{m=1}^{\infty} \left(\bar{A}_m + \frac{2 p_0 a^4 (-1)^{m+1}}{D\pi^5 m^5} \right) \sin \frac{m\pi x}{a} \qquad (66)$$

where $A_m = \bar{A}_m \frac{p_0 a^4}{D}$ (67)

and $B_m = \bar{B}_m \frac{p_0 a^4}{D}$ (68)

For square Kirchhoff plate under hydrostatic load, the deflection along the *x*-axis is obtained as:

$$w(x, y = 0) = \frac{p_0 a^4}{D} \left(2.055 \sin \frac{\pi x}{a} - 0.177 \sin \frac{2\pi x}{a} + 0.025 \sin \frac{3\pi x}{a} \ldots \right) \times 10^{-3} \qquad (69)$$

The deflection at the center ($x = a/2$, $y = 0$) of a square Kirchhoff plate under hydrostatic load is then found as:

$$w\left(x = a/2,\ y = 0\right) = 2.03 \times 10^{-3} \frac{p_0 a^4}{D} \qquad (70)$$

By using the calculus of maxima and minima on Equation (69), we find that maximum deflection occurs at the point given by:

$$\frac{\partial w}{\partial x}(x,\ y = 0) = 0 \qquad (71)$$

Using Equation (71) on Equation (69) we obtain:

$x = 0.557a$ (72)

The maximum deflection of a square Kirchhoff plate under hydrostatic load is thus obtained as:

$$w_{\max} = w\left(x = 0.557a,\ y = 0\right) = 2.06 \times 10^{-3} \frac{p_0 a^4}{D} \qquad (73)$$

4.8 Bending Moment Distribution:

The bending moment distribution is obtained using the bending moment deflection relations:

$$M_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \qquad (74)$$

$$M_{yy} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \qquad (75)$$

$$M_{xx} = -D\left\{ \sum_{m=1}^{\infty} -\left(A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} + \frac{2 p_0 a^4}{D\pi^5} \frac{(-1)^{m+1}}{m^5} \right) \left(\frac{m\pi}{a} \right)^2 \sin \frac{m\pi x}{a} + \mu \sum_{m=1}^{\infty} A_m \left(\frac{m\pi}{a} \right)^2 \cosh \frac{m\pi y}{a} + B_m \frac{m\pi}{a} \left(\frac{m\pi}{a} \cosh \frac{m\pi y}{a} + \frac{m\pi}{a} \left(\cosh \frac{m\pi y}{a} + \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \right) \right\} \sin \frac{m\pi x}{a} \qquad (76)$$

$$M_{yy} = -D \sum_{m=1}^{\infty} \left\{ A_m \left(\frac{m\pi}{a} \right)^2 \cosh \frac{m\pi y}{a} + B_m \frac{m\pi}{a} \left(\frac{m\pi}{a} \cosh \frac{m\pi y}{a} + \frac{m\pi}{a} \left(\cosh \frac{m\pi y}{a} + \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \right) \sin \frac{m\pi x}{a} - \mu \sum_{m=1}^{\infty} \left(A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} + \frac{2 p_0 a^4}{D\pi^5} \frac{(-1)^{m+1}}{m^5} \right) \right\} \left(\frac{m\pi}{a} \right)^2 \sin \frac{m\pi x}{a} \qquad (77)$$

At the center of the plate $x = a/2$, $y = 0$,

$$M_{xx}\left(a/2,\ 0\right) = -D \sum_{m=1}^{\infty} -\left(A_m + \frac{2 p_0 a^4}{D\pi^5} \frac{(-1)^{m+1}}{m^5} \right) \left(\frac{m\pi}{a} \right)^2 \sin \frac{m\pi}{2} + \mu \left(A_m \left(\frac{m\pi}{a} \right)^2 + B_m \frac{m\pi}{a} \left(\frac{m\pi}{a} + \frac{m\pi}{a} \right) \right) \sin \frac{m\pi}{2} \qquad (78)$$

$$= -D \sum_{m=1}^{\infty} \left[-\left(A_m + \frac{2 p_0 a^4}{D\pi^5} \frac{(-1)^{m+1}}{m^5} \right) + \mu(A_m + 2B_m) \right] \left(\frac{m\pi}{a} \right)^2 \sin \frac{m\pi}{2} \qquad (79)$$

$$M_{xx}\left(a/2,\ 0\right) = D \sum_{m=1}^{\infty} \left((1-\mu)A_m - 2\mu B_m + \frac{2 p_0 a^4 (-1)^{m+1}}{D\pi^5 m^5} \right) \left(\frac{m\pi}{a} \right)^2 \sin \frac{m\pi}{2} \qquad (80)$$

Similarly,

$$M_{yy}\left(a/2,\ 0\right) = -D \sum_{m=1}^{\infty} \left(A_m \left(\frac{m\pi}{a} \right)^2 + B_m \frac{m\pi}{a} \left(\frac{m\pi}{a} + \frac{m\pi}{a} \right) \right) \sin \frac{m\pi}{2}$$

$$-\mu\sum_{m=1}^{\infty}\left(A_m+\frac{2p_0a^4(-1)^{m+1}}{D\pi^5m^5}\right)\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi}{2} \quad (81)$$

$$=-D\sum_{m=1}^{\infty}\left(A_m+2B_m-\mu A_m-\frac{2p_0a^4(-1)^{m+1}}{D\pi^5m^5}\right)\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi}{2} \quad (82)$$

$$=-D\sum_{m=1}^{\infty}\left((1-\mu)A_m+2B_m-\frac{2p_0a^4(-1)^{m+1}}{D\pi^5m^5}\right)\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi}{2} \quad (83)$$

$$=D\sum_{m=1}^{\infty}\left((\mu-1)A_m-2B_m+\frac{2p_0a^4(-1)^{m+1}}{D\pi^5m^5}\right)\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi}{2} \quad (84)$$

The equations for the deflection along the *x* axis, Equation (66), and bending moments Equations (80) and (84) at the center of a rectangular Kirchhoff plate under hydrostatic load are then used to determine the values for deflection and bending moments for varying values of the plate aspect ratios *b*/*a*. The deflections and bending moments were then expressed in terms of deflection coefficients, α, and bending moment coefficients, β_{xx}, β_{yy} and presented in Table 1, where

$$w\left(x=a/2,\ 0\right)=\alpha p_0\, a^4/D \quad (85)$$

$$M_{xx}\left(a/2,\ 0\right)=\beta_{xx}p_0a^2 \quad (86)$$

$$M_{yy}\left(a/2,\ 0\right)=\beta_{yy}p_0a^2 \quad (87)$$

Table 1: Deflection and bending moment coefficients at the center of rectangular Kirchhoff plates under hydrostatic load distribution $p(x)=p_0\,x/a$

b/a	$w=\alpha p_0\, a^4/D\times10^{-3}$	$M_{xx}=\beta_{xx}p_0a^2$	$M_{yy}=\beta_{yy}p_0a^2$
1	2.03	0.0239	0.0239
1.1	2.43	0.0276	0.0247
1.2	2.82	0.0313	0.0250
1.3	3.19	0.0346	0.0252
1.4	3.53	0.0376	0.0253
1.5	3.86	0.0406	0.0249
1.6	4.15	0.0431	0.0246
1.7	4.41	0.0454	0.0243
1.8	4.65	0.0474	0.0239
1.9	4.87	0.0492	0.0235
2	5.06	0.0508	0.02322
3	6.12	0.0594	0.0202
4	6.41	0.0617	0.0192
5	6.48	0.0623	0.0187
∞	6.51	0.0625	0.0187

5. DISCUSSIONS

The Kantorovich-Vlasov method has been successfully applied to the bending analysis of simply supported rectangular Kirchhoff plates under linearly distributed transverse load of intensity $p(x) = p_0x/a$ applied over the entire plate domain. Vlasov's method was used to obtain the displacement shape function of the plate in the *x*-direction as Equation (28). Kantorovich variational method was then applied to determine the deflection function which minimized the total potential energy functional Π for the Kirchhoff plate presented in Equation (29). In seeking to extremize the total potential energy functional, Kantorovich method was adopted and the unknown plate deflection function was sought in the form expressed as Equation (31) which contained unknown functions $g_m(y)$ of the *y*-coordinate variable. This resulted in the total potential energy functional expressed by Equation (35), which depends upon *y*, $g_m(y)$ and $g''_m(y)$. Euler-Lagrange differential equation was then applied to obtain the conditions for extremum of the total potential energy functional as Equation (39), a fourth order linear ordinary differential equation in terms of $g_m(y)$. The corresponding Euler-Lagrange differential equation, Equation (39), was solved using the method of undetermined parameters, to obtain the homogeneous solution as Equation (47) and the particular solution as Equation (54). Symmetry was applied to obtain the general solution for the deflection from the linearity principle as Equation (55). The boundary conditions along the edges $y=\pm b/2$ were used to obtain the unknown integration constants as Equations (62) and (63). Thus the deflection was completely determined as Equation (64), which was used to obtain the deflection of square plates along the *x*-axis as Equation (69) and the center deflection as Equation (70). The maximum deflection was found to occur at $x = 0.557a$, and was found to be given by Equation (73). The bending moment distributions M_{xx}, M_{yy} over the plate domain were obtained as Equations (76) and (77). These were used to obtain the bending moments at the center of the plate as Equations (80) and (84). The bending moment Equations (80) and (84) were then used to compute the bending moments coefficients β_{xx} and β_{yy} for the plate center which are shown in Table 1. A comparison of the deflection coefficients and bending moment coefficients presented in Table 1 shows that they are the same as those presented by Timoshenko and Woinowsky-Krieger for the same problem of simply supported plates solved using Levy's method.

6. CONCLUSIONS

From the study, the following conclusions are made:

(i) the Kantorovich-Vlasov method can be successfully applied to the solution of Kirchhoff plate flexure problems for simply supported edges and static transverse loads.

(ii) the solution obtained for deflection function is a single trigonometric series containing hyperbolic functions.

(iii) the solutions obtained for the bending moment distributions M_{xx}, M_{yy} are single trigonometric series containing hyperbolic functions.

(iv) the solution obtained for deflection at the center of the rectangular Kirchhoff plate under hydrostatic load distribution is a rapidly convergent series; and convergent results to the exact solutions for center deflections were obtained using a few terms of the series.

(v) the solutions obtained for bending moments (M_{xx} and M_{yy}) at the center of the plate are rapidly convergent series; and convergent results to the exact solutions for bending moments at the center were obtained using a few terms of the series.

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