KANTOROVICH METHOD FOR THE DETERMINATION OF EIGEN FREQUENCIES OF THIN RECTANGULAR PLATES

*Ike, C.C.[1] and Nwoji, C.U.[2]

1 Department of Civil Engineering, Enugu State University of Science & Technology (ESUT). Enugu State, Nigeria.
2 Department of Civil Engineering, University of Nigeria Nsukka (UNN). Enugu State, Nigeria.

* Author for Correspondence: Ike, C.C; Email: charles.ike@esut-edu.ng

ABSTRACT

In this work, the Kantorovich variational method was applied to determine the natural frequencies of vibration of thin rectangular plates with two opposite edges clamped and the other two edges simply supported. Kirchhoff-Love theory of plates was used to describe the governing differential equation of the plate under dynamic loads. The plate considered was assumed to be made of homogeneous, isotropic material. Following, Kantorovich methodology, the deflection function was assumed to be a linear combination of the product of coordinate (basis) functions that satisfy *apriori* the deflection boundary conditions on the simply supported edges x = 0, x = a; and unknown functions $Y_m(y)$ of the space coordinate of y which will be determined using variational calculus. The variation integral statement of the plate was then obtained using Galerkin's procedure. The variational integral obtained was found to reduce to fourth order ordinary differential equations in $Y_m(y)$. This was solved, subject to the deflection boundary conditions of the clamped ends y = 0, y = b to obtain the algebraic eigen value – eigen vector problem which was solved to obtain the characteristic frequency equation. The roots yielded the natural (eigen) frequencies of the plate. The natural frequencies obtained were compared with natural frequencies obtained from the researchers in literature who used Galerkin-Vlasov, Levy, Finite Difference and Rayleigh methods. It was found that Kantorovich results agreed closely with the Galerkin-Vlasov and Levy methods, confirming the effectiveness of the method.

Keywords: *Kantorovich variational method, Kirchhoff-Love plate, coordinate (basis) function, algebraic eigen value, eigen vector problem, variational integral, Eigen frequencies.*

1. INTRODUCTION

Plates are three dimensional structural elements characterized by length, width and a much smaller transverse dimension called the thickness. They are usually flat, and when curved, they are called shells (Timoshenko and Woinowsky-Krieger, 1959). They have extensive applications in civil, mechanical, aeronautical, marine, naval, structural and geotechnical engineering as bridge decks, retaining walls, aircraft panels, spacecraft panels, machine parts, etc. They can be made of reinforced concrete or metal and can be shaped as rectangular, circular, elliptical, skewed, and trapezoidal (Mansfield, 1964; Ugural, 1999). Plates are generally classified as thin plates $8 < a/h < 100$, moderately thick plates and thick plates $\left(a/h \leq 8 - 10\right)$ depending on the ratio of a/h where h is the plate thickness and a is the least in plane dimension of the plates (Ventsel and Krauthammer, 2001). Plates can be subject to static or dynamic loads/forces; and can be subjected to inplane compression which causes buckling (Chakraverty, 2009; Nowacki, 1963). The focus of this paper is to study the natural frequencies of vibration of thin plates with two opposite edges clamped and the other two opposite edges on simple supports. There are several types of plate theories. The basic idea of plate theories is to reduce the three dimensional plate problem defined by the equations of the theory of elasticity to two dimensional approximations. This reduction of dimension is accomplished by integrating out one of the dimensions, usually the transverse dimension, z, by expressing stresses using resultant forces. The following plate theories exist: Kirchhoff-Love plate theory, Mindlin plate theory (Bletzinger Kai Ume, 2008), Reissner plate theory, Levinson (1980) plate theory, Reddy plate theory (Wang, et al, 2000), Von Karman plate theory (Large deflection plate theory), Refined plate theories (Shimpi, 2002; Shimpi and Patel, 2006; Suetake, 2006, Lo et al, 1977, Szilard, 2004).

One of the steps in performing a dynamic analysis of plates is determining the natural frequencies and mode shapes of the plate. These characterize the basic dynamic behaviour of the plate and are an indication of how the plate would respond to dynamic loading. The natural frequencies of a plate are the frequencies at which the plate tends to vibrate if it is subjected to an excitation. Other commonly used terms for natural frequency are characteristic frequency, eigen frequency, fundamental frequency, resonant frequency, normal frequency and resonance frequency. The deformed shape of the plate at a specific natural frequency of vibration is termed the normal mode of vibration. Some other terms used for normal mode are mode shape, characteristic shape, and fundamental shape. Each mode shape is associated with a particular natural frequency (Yang et al, 2014; Navita, 1979; Young, 1950; Bhardwaj et al, 2012; Khare and Mittal, 2015). Exact solutions of the free transverse vibration of rectangular Kirchhoff-Love plates have been found for just a few types of boundary conditions. For more difficult cases, approximate methods are used for calculating the natural frequencies and mode slopes (Reed, 1965). The exact methods used for dynamic analysis are the Navier double trigonometric series method which is suitable for plates with all edges simply supported and the Levy single trigonometric series method, which is suitable for rectangular plates with mixed support conditions. The approximate methods that have been commonly used include Finite Difference method, Finite Element method, Ritz variational method, Galerkin variational method (Balasubramanian, 2011) Collocation methods, Weighted Residual methods, and Integral transform methods (Tian et al, 1958).

Xing and Liu (2009) used the method of separation of variables to solve for the exact solutions of free vibrations of thin orthotropic rectangular plates with all combinations of simply supported and clamped edge boundary conditions. Phamová and Vampola (2016) determined free flexural vibration modes and eigen frequencies of a thin plate with general boundary conditions by transforming the governing partial differential equation into two ordinary differential equations that can be easier to solve. Bercin (1996) used the Kantorovich method to obtain the natural frequencies of vibration of clamped orthotropic plates.

Fallah and Khakbaz (2017) have successfully applied the extended Kantorovich method based on the first order shear deformation plate theory to solve the bending problem of functionally graded annular sector plates with arbitrary boundary conditions subjected to both uniformly distributed and non-uniformly distributed loads. They employed two approaches, using the functional of the problem, and the weighted integral form of the governing differential equations and solved the resulting ordinary differential equations.

2. METHODOLOGY

In the Kantorovich method, the displacement field is described by the functions given as the products of assumed shape (coordinate) functions and of unknown functions, which then reduce the governing partial differential equation of equilibrium to ordinary differential equations in terms of the unknown functions (Lee, 2009). The assumed displacement coordinate functions must satisfy the boundary conditions at two parallel edges in one coordinate direction. The unknown functions are determined by enforcement of boundary conditions in the other two directions on the solutions to the resulting ordinary differential equation (Lee, 2009). The governing partial differential equation of Kirchhoff-Love plates undergoing vibration is

$$D\nabla^4 w(x,y,t) + \rho h w_{tt}(x,y,t) = p_z(x,y,t) \qquad (1)$$

where $w(x,y,t)$ is the dynamic deflection, ρ is the density of the plate, h is the plate thickness, p_z is the external time dependent dynamic load, D is the plate rigidity.

For free harmonic vibrations, the displacement response $w(x,y,t)$ would vary harmonically with time, such that $w(x,y,t) \equiv W(x,y)\exp(i\omega_{mn}t)$, and there will be no excitation force, where ω_{mn} is the natural frequencies of harmonic vibration.

$$p_z \equiv 0 \qquad (2)$$

and the equation simplifies to

$$\left((D\nabla^4 W(x,y) - \rho h\omega_{mn}^2 W(x,y)\right)e^{i\omega_{nt}t} = 0 \qquad (3)$$

or

$$\nabla^4 W - \frac{\rho h\omega_{mn}^2}{D}W = 0 \qquad (4)$$

Following Kantorovich-Galerkin's variational method (Kantorovich and Krylov, 1958)

$$W(x,y) = \sum_{m=1}^{\infty} Y_m(y)\sin\frac{m\pi x}{a} \qquad (5)$$

for plates simply supported on edges $x = 0$, $x = a$

The variational integral statement becomes the double integral over the plate domain given by:

$$\int_0^b\int_0^a \nabla^4\left(\sum_{m=1}^{\infty} Y_m(y)\sin\frac{m\pi x}{a}\right) - \frac{\rho h\omega_{mn}^2}{D}\left(\sum_{m=1}^{\infty} Y_m(y)\sin\frac{m\pi x}{a}\right)\sin\frac{m'\pi x}{a}\,dxdy = 0 \qquad (6)$$

The variational integral is solved to obtain $Y_m(y)$.

3. KANTOROVICH METHOD FOR THE NATURAL FREQUENCIES OF VIBRATION OF THIN RECTANGULAR PLATES WITH TWO OPPOSITE EDGES CLAMPED AND THE OTHER TWO OPPOSITE EDGES SIMPLY SUPPORTED

The thin rectangular plate considered is shown in Figure 1. The plate has two opposite edges ($y = 0$, and $y = b$) clamped and the other edges $x = 0$, $x = a$ simply supported

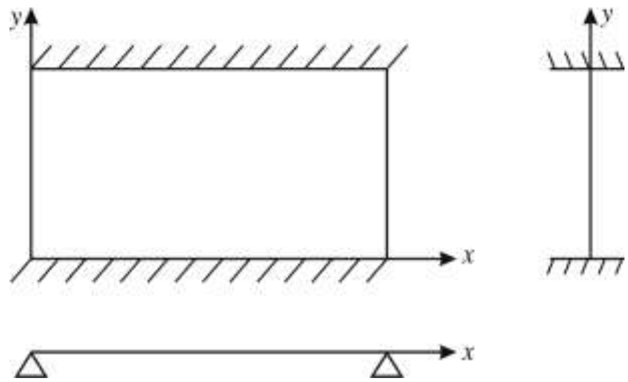

Figure 1: *Thin rectangular plate with opposite edges clamped and the other edges simply supported*

The governing partial differential equation for Kirchhoff-Love plates subjected to dynamic loads is given by Equation (1).

For free vibrations, there is no externally applied excitation, $p(x, y, t) \equiv 0$ and the governing equation simplifies to

$$D\nabla^4 w + \rho h w_{tt} = 0 \quad (7)$$

The plate is clamped at $y = 0$, $y = b$ and simply supported at $x = 0$, $x = a$. The deflection and force boundary conditions are:

$$\begin{aligned} &w(x=0, y, t) = 0; \qquad w(x=a, y, t) = 0 \\ &w,_{xx}(x=0, y, t) = 0; \quad w,_{xx}(x=a, y, t) = 0 \\ &w(x, y=0, t) = 0; \qquad w(x, y=b, t) = 0 \\ &w,_{y}(x, y=0, t) = 0 \qquad w,_{y}(x, y=b, t) = 0 \end{aligned} \quad (8)$$

where $w,_y$ represents the derivative of $w(x, y)$ with respect to *y*.

For harmonic vibrations, we assume

$$w(x, y, t) = W(x, y)e^{i\omega t} \quad (9)$$

where ω_{mn} is the frequency, then from Equation (50), the variational integral becomes:

$$\int_0^a\int_0^b \left(D\nabla^4 W(x, y)e^{i\omega t} - \rho h\omega^2 W(x, y)e^{i\omega t}\right)\sin\frac{m'\pi x}{a}dxdy = 0 \quad (10)$$

$$\iint\left(\left(\nabla^4\sum_{m=1}^{\infty}Y_m(y)\sin\frac{m\pi x}{a}\right) - \frac{\rho h\omega^2}{D}\left(\sum_{m=1}^{\infty}Y_m(y)\sin\frac{m\pi x}{a}\right)\right)e^{i\omega t}\sin\frac{m'\pi x}{a}dxdy = 0 \quad (11)$$

$$\sum_{m=1}^{\infty}\int_0^a\int_0^b\left(\left(\nabla^4 - \lambda^4\right)Y_m(y)\sin\frac{m\pi x}{a}\right)\sin\frac{m'\pi x}{a}dxdy = 0 \quad (12)$$

$$\sum_{m=1}^{\infty}\iint\left\{\left(\frac{m\pi}{a}\right)^4\sin\frac{m\pi x}{a}Y_m(y) - 2\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi x}{a}Y''_m(y) + \sin\frac{m\pi x}{a}Y_m^{iv}(y) - \lambda^4\sin\frac{m\pi x}{a}Y_m(y)\right\}\sin\frac{m'\pi x}{a}dxdy\exp i\omega t = 0 \quad (13)$$

$$\sum_{m=1}^{\infty}\int_0^a\int_0^b\left(Y_m^{iv}(y) - 2\left(\frac{m\pi}{a}\right)^2 Y''_m + \left(\left(\frac{m\pi}{a}\right)^4 - \lambda^4\right)Y_m(y)\right) \times \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}dxdy = 0 \quad (14)$$

Considering the orthogonal property of the function $\sin\frac{m\pi x}{a}$, we have

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}dx = \begin{cases} 0 \text{ if } m \neq m' \\ \frac{a}{2} \text{ if } m = m' \end{cases} \quad (15)$$

$$\sum_{m=1}^{\infty}\int_0^b\left(Y_m^{iv}(y) - 2\left(\frac{m\pi}{a}\right)^2 Y''_m(y) + \left(\left(\frac{m\pi}{a}\right)^4 - \lambda^4\right)Y_m(y)\right)dy = 0 \quad (16)$$

The characteristic equation becomes the fourth order ordinary differential equation in $Y_m(y)$

$$Y_m^{iv}(y) - 2\left(\frac{m\pi}{a}\right)^2 Y''_m(y) + \left(\frac{m^4\pi^4}{a^4} - \lambda^4\right)Y_m(y) = 0 \quad (17)$$

We solve for $Y_m(y)$ using the method of trial functions. Let us assume a solution for $Y_m(y)$ in the form

$$Y_m(y) = A\exp(s\,y) \quad (18)$$

where A is a constant (parameter) we wish to find.

Then by substitution into the characteristic equation, we obtain:

$$\left(s^4 - 2\left(\frac{m^2\pi^2}{a^2}\right)s^2 + \left(\frac{m^4\pi^4}{a^4} - \lambda^4\right)\right)Ae^{sy} = 0 \quad (19)$$

For non trivial solutions,

$$Y_m(y) = Ae^{sy} \neq 0 \quad (20)$$

Hence,

$$s^4 - 2\left(\frac{m^2\pi^2}{a^2}\right)s^2 + \left(\frac{m^4\pi^4}{a^4} - \lambda^4\right) = 0 \quad (21)$$

$$\left(s^2 - \frac{m^2\pi^2}{a^2}\right)^2 - \lambda^4 = 0 \quad (22)$$

$$\left(s^2 - \frac{m^2\pi^2}{a^2} + \lambda^2\right)\left(s^2 + \frac{m^2\pi^2}{a^2} - \lambda^2\right) = 0 \quad (23)$$

Thus, $s^2 = \left(\lambda^2 + \frac{m^2\pi^2}{a^2}\right)$ (24)

or $s^2 = \left(\frac{m^2\pi^2}{a^2} - \lambda^2\right)$ (25)

$$s^2 = (-1)\left(\lambda^2 - \frac{m^2\pi^2}{a^2}\right) \quad (26)$$

$$s_{1,2} = \alpha = \pm\sqrt{\left(\lambda^2 + \frac{m^2\pi^2}{a^2}\right)} \quad (27)$$

$$s_{2,3} = \beta = \pm i\sqrt{\left(\lambda^2 - \frac{m^2\pi^2}{a^2}\right)} \quad (28)$$

The solution for $Y_m(y)$ then becomes:

$$Y_m(y) = c_{1m}\cosh\alpha y + c_{2m}\sinh\alpha y + c_{3m}\cos\beta y + c_{4m}\sin\beta y \quad (29)$$

Using Equation (5), the deflection becomes:

$$W(x,y) = \sum_{m=1}^{\infty}(c_{1m}\cosh\alpha y + c_{2m}\sinh\alpha y + c_{3m}\cos\beta y + c_{4m}\sin\beta y)\sin\frac{m\pi x}{a} \quad (30)$$

$W(x, y)$ is required to satisfy the boundary conditions

$$w(x, y=0) = 0, \quad w(x, y=b) = 0 \quad (31)$$

$$w,_y(x, y=0) = 0, \quad w,_y(x, y=b) = 0 \quad (32)$$

where the comma, after w denotes partial derivative with respect to y.

$$\frac{\partial w}{\partial y} = \sum_{m=1}^{\infty}(c_{1m}\alpha\sinh\alpha y + c_{2m}\alpha\cosh\alpha y - c_{3m}\sin\beta y + c_{4m}\cos\beta y)\sin\frac{m\pi x}{a} \quad (33)$$

$$w(x, y=0) = c_{1m} + c_{3m} = 0 \quad (34)$$

$$w(x, y=b) = c_{1m}\cosh\alpha b + c_{2m}\sinh\alpha b + c_{3m}\cos\beta b + c_{4m}\sin\beta b = 0 \quad (35)$$

$$\frac{\partial w}{\partial y}(x, y=0) = c_{2m}\alpha + c_{4m}\beta = 0 \quad (36)$$

$$\frac{\partial w}{\partial y}(x, y=b) = c_{1m}\alpha\sinh\alpha b + c_{2m}\alpha\cosh\alpha b - c_{3m}\beta\sin\beta b + c_{4m}\beta\cos\beta b = 0 \quad (37)$$

In the matrix form, Equations (34-37) become the system of homogeneous equations:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \cosh\alpha b & \sinh\alpha b & \cos\beta b & \sin\beta b \\ \alpha\sinh\alpha b & \alpha\cosh\alpha b & -\beta\sin\beta b & \beta\cos\beta b \end{vmatrix}\begin{vmatrix} c_{1m} \\ c_{2m} \\ c_{3m} \\ c_{4m}\end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0\end{vmatrix} \quad (38)$$

This is now a homogeneous equation. For non trivial solutions, the determinant of the coefficient matrix is equated to zero, to obtain the characteristic equation:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \cosh\alpha b & \sinh\alpha b & \cos\beta b & \sin\beta b \\ \alpha\sinh\alpha b & \alpha\cosh\alpha b & -\beta\sin\beta b & \beta\cos\beta b \end{vmatrix} = 0 \quad (39)$$

Expanding the determinant, we have,

$$\begin{vmatrix} \alpha & 0 & \beta \\ \sinh\alpha b & \cos\beta b & \sin\beta b \\ \alpha\cosh\alpha b & -\beta\sinh\beta b & \beta\cos\beta b \end{vmatrix} + \begin{vmatrix} 0 & \alpha & \beta \\ \cosh\alpha b & \sinh\alpha b & \sin\beta b \\ \alpha\sinh\alpha b & \alpha\cosh\alpha b & \beta\cos\beta b \end{vmatrix} = 0 \quad (40)$$

Expanding, we obtain after simplification,

$$2\alpha\beta(1 - \cosh\alpha b\cos\beta b) + (\alpha^2 - \beta^2)\sinh\alpha b\sin\beta b = 0 \quad (41)$$

or $2\alpha\beta(\cosh\alpha b\cos\beta b - 1) + (\beta^2 - \alpha^2)\sinh\alpha b\sin\beta b = 0$ (42)

This is the characteristic frequency equation in α and β. It is a transcendental equation, which is solved to find α and β.

Solving, for $m = 1$, $n = 1$

$$\alpha^2 = \frac{38.8156}{a^2} \quad (43)$$

$$\alpha = \frac{6.2302}{a} \quad (44)$$

$$\beta^2 = \frac{19.0764}{a^2} \quad (45)$$

$$\beta = \frac{4.36765}{a} \quad (46)$$

For $m = 2$, $n = 1$

$$\alpha^2 = \frac{64.6126}{a^2} \quad (47)$$

$$\alpha = \frac{8.0382}{a} \quad (48)$$

$$\beta^2 = \frac{44.8734}{a^2} \quad (49)$$

$$\beta = \frac{6.69876}{a} \quad (50)$$

For $m = 1$, $n = 2$

$$\alpha^2 = \frac{79.1966}{a^2} \quad (51)$$

$$\alpha = \frac{8.89925}{a} \quad (52)$$

$$\beta^2 = \frac{59.4574}{a^2} \quad (53)$$

$$\beta = \frac{7.71086}{a} \quad (54)$$

So, for m = 1, n = 1,

$$\lambda_{11}^2 = \frac{28.946}{a^2} \quad (55)$$

$$\lambda_{11} = \frac{5.38015}{a} \quad (56)$$

$$\lambda_{mn} = \frac{\omega_{mn}^2 \rho h}{D} \quad (57)$$

$$\lambda_{11}^4 = \left(\frac{5.38015}{a}\right)^4 = \frac{\omega_{11}^2 \rho h}{D} \quad (58)$$

$$\omega_{11}^2 = \frac{D}{\rho h}\left(\frac{5.38015}{a}\right)^4 \quad (59)$$

$$\omega_{11} = \frac{28.946}{a^2}\sqrt{\frac{D}{\rho h}} \quad (60)$$

Similarly, $$\omega_{mn} = \lambda_{mn}^2 \sqrt{\frac{D}{\rho h}} \quad (61)$$

$$\omega_{21} = \frac{54.743}{a^2}\sqrt{\frac{D}{\rho h}} \quad (62)$$

$$\omega_{12} = \frac{69.327}{a^2}\sqrt{\frac{D}{\rho h}} \quad (63)$$

$$\omega_{22} = \frac{97}{a^2}\sqrt{\frac{D}{\rho h}} \quad (64)$$

4. RESULTS, DISCUSSIONS AND CONCLUSIONS

In this paper, the Kantorovich variational method has been successfully applied to determine the natural frequencies of vibration of Kirchhoff-Love plates with two opposite edges (y = 0, y = b) clamped, and the other two edges (x = 0, x = a) simply supported. The plate was assumed to be made of homogeneous, isotropic material. The Kirchhoff-Love differential equation for plate flexure under free vibration loading conditions was written in variational form, based on the Kantorovich method. Hence, the deflection basis function was assumed to be a linear combination of m coordinate basis functions in the x-direction that satisfy apriori the deformation boundary conditions on the edges x = 0 and x = a; and m unknown functions $Y_m(y)$ of y in the y-direction. The variational integral formulated based on Kantorovich methodology was obtained as Equation (11). The Kantorovich variational equation was further simplified to yield Equation (16) and subsequently, the fourth order ordinary differential Equation (17). This was solved using the method of trial functions (also called the method of undetermined parameters) to obtain the general solution for $Y_m(y)$ presented in Equation (29). Thus, the general solution for the deflection modal shape function was obtained as Equation (30). The boundary conditions on y = 0, y = b were enforced on $Y_m(y)$ to obtain the system of homogeneous equations given as Equation (38) in terms of the four constants of integration. For non trivial solutions, the homogeneous equation yielded the characteristic frequency equation shown in Equation (39). Expansion and simplification of the characteristic frequency equation yielded Equations (41) and (42), which are transcendental equations in terms of α and β. The transcendental equation was solved to find the roots α, and β for various integral values of m, and n. Since α and β were defined in terms of the natural frequencies of the plate, the natural frequencies were thus obtained. The natural frequencies obtained are tabulated in Table 1, which also shows the natural frequencies of Kirchhoff-Love plates with the same edge support conditions obtained by other researchers. A comparison of the natural frequencies obtained shows that the Kantorovich method yielded natural frequencies which were identical with the natural frequencies obtained using the Galerkin-Vlasov method and the Levy Method. The Kantorovich variational method is thus an effective and efficient method for the free vibration analysis of eigen frequencies of

Kirchhoff-Love plates with two opposite edges clamped and the other two edges simply supported.

Table 1: Natural frequencies of free vibrations of square flat plate clamped on two opposite edges and simply supported on the other two

$$\omega_{mm} = \frac{\lambda_{mn}^2}{a^2}\sqrt{\frac{D}{\rho h}}$$

Eigen values	λ_{11}^2	λ_{12}^2	λ_{13}^2	λ_{21}^2	λ_{22}^2
Kantorovich -Galerkin (Present study)	28.946	69.327		54.743	97
Galerkin-Vlasov	28.944	70.11	123.16	54.93	97.07
Levy method	28.946	69.32	129.086	54.743	94.584
Finite Difference method	28.974				
Rayleigh method	29.57				

5. REFERENCES

Balasubramanian, Ashwin. (2011). Plate Analysis with Different Geometrics and Arbitrary Boundary Conditions. M.Sc Thesis in Mechanical Engineering, Faculty of the Graduate School, University of Texas at Arlington.

Bercin, A.N. (1996). Free vibration solution for clamped orthotropic plates using the Kantorovich method. Journal of Sound and Vibration 19 6(2) pp. 243 – 247.

Bhardwaj N., Gupta A.P., Choong K.K., Wang C.M., and Ohmori H. (2012). Transverse vibrations of clamped and simply supported circular plates with two dimensional thickness vibration. Shock and Vibration, Vol. 19 (2012) pp 273-285 Do/10,3233/SAV/-2011-0630 105 Press., 2012.

Bletzinger Kai-Ume (2008). Theory of Plates Part II: Plates in Bending Lecture Notes Winter Semester 2012/2013 Lehrstuhal fur static technisache universitat Muchen, October 2008 http//www.saribd.com/doc/112345132/theory of plates – part2, 2008.

Chakraverty S. (2009). Vibration of Plates. Boca Raton CRC Press Taylor and Francis Group.

Fallah, F. and Khakbaz, A. (2017). On an extended Kantorovich method for the mechanical behavior of functionally graded solid/annular sector plates with various boundary conditions. Acta Mechanica, pp. 1 – 20, Springer Vienna doi: 10.1007/s00707 – 017 – 1851 – 2.

Kantorovich L.V., and Krylov V.I. (1958). Approximate Methods of Higher Analysis. John Wiley and Sons. New York.

Khare, S. and Mittal, N.D. (2015). Free vibration analysis of thin circular and annular plate with general boundary conditions. Engineering solid mechanics 3, (2015) pp 245-252, 2015.

Lee, U. (2009) Spectral Element Method in Structural Dynamics. John Wiley and Sons (Asia) Pte Ltd Singapore. https//books.google.com/books?isbn=0470823755.

Levinson M. (1980) An accurate simply theory of the statics and dynamics of elastic plates. Mech Res. Common 7, pp 343-350, 1980.

Lo K.H., Christenson R.M., and Wu E.M. (1977). A high order theory of plate deformation: Part 1. Homogeneous plates J. Applied Mechanics, Vol. 44, (1977) pp 663-668, 1977.

Mansfield E.H. (1964). The Bending and Stretching of Plates. Macmillan, New York,

Navita Y. (1979). Free Vibration of Elastic Plates with Various Shapes and Boundary Conditions. PhD Thesis, Graduate School of Engineering, Hokkaido University.

Nowacki W. (1963). Dynamics of Elastic Systems. John Wiley and Sons, New York.

Phamova, L. and Vampola, T. (2016). Vibration modes of a single plate with general boundary conditions. Applied and Computational Mechanics 10 (2016) pp. 49 – 56.

Reed R.E. (1965). Comparison of Methods in Calculating Frequencies of Corner-Supported Rectangular Plates. Ames Research Center, Moffat field California, National Aeronautics and Space Administration Technical Note, Washinton D.C.

Reissner E. (1983). A twelfth order theory of transverse bending of transversely isotropic plates. Z Angew Math Mech (ZAMM), 63, (1983) pp 285-289, 1983.

Shimpi R.P and Patel H.G. (2006). A two variable refined plate theory for orthotropic plate analysis. International Journal of Solids and Structures, El servier, Vol 43, Issues 22-23, November 2006, pp 6783-6799, 2006.

Shimpi R.P. (2002). Refined plate theory and its variants. AIAA Journal vol 40, No 1 January 2002.

Suetake Y. (2006). Plate bending analysis by using a modified plate theory. CMES, Vol 11 No.3, Tech Suavie Press pp 103-110, 2006.

Szilard R. (2004) Theories and Applications of Plate Analysis, Classical Numerical and Engineering Methods. JohnWiley and Sons Inc. New Jerssy pp 139-140, 2004.

Tian Bu., Zhong Yi, Li R. (2011). Analytic Bending Solutions of Rectangular Cantilver Thin Plates. Archives of Civil and Mechanical Engineering, Vol. XI No 4, pp 1043-1052, 2011.

Timoshenko S., and Woinowsky-Krieger S. (1959). Theory of Plates and Shells, 2nd Edition. McGraw Hill, New York.

Ugural A.C. (1999). Stresses in Plates and Shells, Second Edition. WCB McGraw Hill Boston.

Ventsel E. and Krauthammer T. (2001). Thin Plates and Shells: Theory, Analysis and Applications. Marcel Dekker Inc. USA.

Wang C.M., Reddy J.N. and Lee K.H. (2000). Shear Deformable Beams and Plates: Relationships with Classical Solutions. Elservier Science Ltd. Amterdam, Oxford U.K. pp 87-110, 2000.

Xiang, Y.F. and Liu, B. (2009). New exact solutions for free vibration of thin orthotropic rectangular plates. Composite Structures 89 (2009) pp. 567 – 574.

Yang Zhong, Xue-Feng Zhao, Heng Liu. (2014). Vibration of plate on foundation with four edges free by finite cosine integral transform method. Latin American Journal of Solids and Structures, Vol. 11, No. 5, Rio de Janiero, October, 2014.

Young D. (1950). Vibration of Rectangular Plates by the Ritz Method. ASME Trans. Journal of Applied Mechanics, Volume 17, No 4, December 4, 1950, pp 448-453, 19